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Flow around a triaxial ellipsoid and a spheroid in a long circular tube

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Abstract

This paper deals with the three-dimensional analysis of ideal fluid flow in a long circular cylinder containing an ellipsoidal obstacle. The center of the ellipsoid coincides with that of the cylinder, and the flow is confined to the space between the ellipsoid and the cylinder when the fluid velocity at the large distance from the ellipsoid is uniform. The equations of the classical theory of fluid dynamics are solved in terms of an unknown function which is then shown to be the solution of a boundary integro-differential equation. An analytical solution of the integro-differential equation is obtained for the moderate values of the radius of the cylinder. The pressure on the ellipsoid is obtained by using Bernoulli's equation and is presented in graphical forms for various values of the radius of the circular tube. The problem of viscous fluid when the ellipsoid reduces to a spheroid is also investigated.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

It has been a long time since the study on the flow around a spherical and non-spherical object in a tube began. The investigations varied from the vortical to irrotational flow and from the inviscid to viscous flow. The problem of determining the distribution of vector potential in a long circular cylinder containing a spherical or a spheroidal obstacle has been investigated by Smithe [1, 2]. The problem of flow around a sphere in a tube has also been investigated by others [3, 4]. However, little attention has been paid to the analytical solution concerning a triaxial ellipsoidal obstacle, as a special case of which the analysis on spheres or spheroids can be dealt. The motion of a viscous liquid past an ellipsoid in an unbounded space was however investigated by Venkates [5].

In more recent years, numerical studies on the motion of an ellipsoid in a circular tube have appeared. Sugihara-Seki [6] studied numerically the motions of an ellipsoidal particle in a tube flow. She used a finite-element method to solve the Stokes equations for flow around a spheroid placed at various positions in the tube. The instantaneous velocity was used to compute the particle trajectories. Swaminathan *et al* [7] have used direct numerical simulations to investigate the motion of an ellipsoid settling in an infinitely long circular tube, under the influence of gravity, at low and intermediate Reynolds numbers. They examined the issue of damping of the oscillatory motion for different cases of particle inertia.

Information on the potential flow around an ellipsoid will be of value to the circumstances that occur in a wind tunnel, to a circular cylindrical flow with bubbles or to an electrical flow in a circular cylindrical conductor with defects that can be approximated by a triaxial ellipsoid.

Applications of the study on such flow can be made in a broad range of biological and engineering fields; examples include flow due to the motion of proteins in various biomedical applications, the motion of red blood cells in narrow capillaries and the transport of encapsuled solid matter in pipelines.

In this paper, we derive the solution of the problem determining the distribution of potential in a long circular cylinder containing a triaxial ellipsoid whose center coincides with that of the cylinder when the flow is uniform at the large distance from the ellipsoid.

In section 2, we investigated an inviscid fluid. By the use of the field equations, and employing Fourier transform, the boundary integro-differential equation is derived in which the unknown function is subsequently solved by the iterative method. In section 3, we investigated a viscous fluid. In section 4, some numerical examples are given and comparisons with the existing published accounts are made.

2. Inviscid flow about a triaxial ellipsoid

In this section we discuss the inviscid flow about a triaxial ellipsoid in a tube. Consider a circular cylinder with a radius, h, having an ellipsoid whose semi-axes are a, b and c. We take the center of the ellipsoid and the cylinder as the origin of the Cartesian coordinates, and, x-, y- and z-axis along semi-axes of the ellipsoid, respectively, then the ellipsoid occupies the region V which is governed by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1$. The surface of the ellipsoid is denoted by S. We shall also use cylindrical coordinates (r, θ, z) which are connected to the Cartesian coordinates by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Let the velocity of the flow at the large distances from the ellipsoid be v_0. Then the suitable form of potential function is

$$\phi(x, y, z) = \frac{\partial}{\partial z}\int_V \frac{f(u, v, w)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w}{\sqrt{(x-u)^2 + (y-v)^2 + (z-w)^2}} + v_0 z + \sum_{m=0}^{\infty}\cos m\theta \int_{-\infty}^{\infty} B_m(\xi) I_m(\xi r)\,\mathrm{e}^{-\mathrm{i}\xi z}\,\mathrm{d}\xi, \tag{2.1}$$

where I_m is the modified Bessel function of the first kind. It can be easily seen that the function ϕ in (2.1) is a harmonic function. The boundary condition satisfied on the wall is

$$\frac{\partial\phi}{\partial r} = 0 \qquad \text{at} \quad r = h. \tag{2.2}$$

Thus we have

$$\frac{\partial^2}{\partial r\partial z}\int_V \frac{f(u, v, w)\,\mathrm{d}\mathbf{v}}{R(\mathbf{x}-\mathbf{u})}\bigg|_{r=h} + \sum_{m=0}^{\infty}\cos m\theta\int_{-\infty}^{\infty} B_m(\xi)\xi I_m'(\xi h)\,\mathrm{e}^{-\mathrm{i}\xi z}\,\mathrm{d}\xi = 0,$$

where

$$R(\mathbf{x}-\mathbf{u}) = \sqrt{(x-u)^2+(y-v)^2+(z-w)^2}$$

and d$\mathbf{v}$ is used for du dv dw. If we take the Fourier transform of the above equation, we get

$$-\sum_{m=0}^{\infty}\cos m\theta B_m(\xi)\xi I_m'(\xi h) = \frac{1}{2\pi}\frac{\partial}{\partial r}\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial z}\int_V \frac{f(u,v,w)\,\mathrm{d}\mathbf{v}}{R(\mathbf{x}-\mathbf{u})}\right)\mathrm{e}^{\mathrm{i}\xi z}\,\mathrm{d}z\bigg|_{r=h}. \quad (2.3)$$

If we once integrate by parts on the right-hand side of the above equation, and make use of the known integral in Erdélyi [8] and Watson [9]

$$\int_0^{\infty}\frac{\cos(\xi z)\,\mathrm{d}z}{\sqrt{(x-u)^2+(y-v)^2+z^2}} = K_0(\xi\sqrt{(x-u)^2+(y-v)^2}), \quad (2.4)$$

we find that

$$\sum_{m=0}^{\infty}\cos m\theta B_m(\xi)\xi I_m'(\xi h) = \frac{\mathrm{i}}{\pi}\int_V f(u,v,w)\,\mathrm{e}^{\mathrm{i}\xi w} \times \frac{\partial}{\partial h}\{K_0(\xi\sqrt{(x-u)^2+(y-v)^2})\}\,\mathrm{d}\mathbf{v}. \quad (2.5)$$

Now

$$K_0(\xi\sqrt{(x-u)^2+(y-v)^2}) = K_0\big(\xi\{r^2+r'^2-2rr'\cos(\theta-\theta')\}^{\frac{1}{2}}\big)$$
$$= I_0(\xi r_<)K_0(\xi r_>) + 2\sum_{m=1}^{\infty}\cos\{m(\theta-\theta')\}I_m(\xi r_<)K_m(\xi r_>), \quad (2.6)$$

where $r_< = \min(r, r')$, and $r_> = \max(r, r')$ and K_0, K_m are the modified Bessel functions of the second kind and we have set

$$u = r'\cos\theta', \qquad v = r'\sin\theta'. \quad (2.7)$$

If we use (2.6) in (2.5), we find that

$$B_m(\xi) = \epsilon_m\frac{\mathrm{i}|\xi|K_m'(|\xi|h)}{\pi I_m'(\xi h)}\int_V f(u,v,w)\,\mathrm{e}^{\mathrm{i}\xi w}I_m(|\xi|r')\cos m\theta'\mathrm{d}\mathbf{v}, \quad (2.8)$$

where

$$\epsilon_m = \begin{cases}1, & \text{if} \quad m=0\\ 2, & \text{otherwise.}\end{cases}$$

In (2.8), the term involving $\sin m\theta'$ vanishes, as $f(u,v,w)$ is an even function of u and v. If we substitute $B_m(\xi)$ from (2.8) into (2.1), we obtain the following form of potential function:

$$\phi = \frac{\partial}{\partial z}\int_V\frac{f(u,v,w)\,\mathrm{d}\mathbf{v}}{R(\mathbf{x}-\mathbf{u})} + v_0 z + \int_V f(u,v,w)K(r,\theta,z;r',\theta',w)\,\mathrm{d}\mathbf{v}, \quad (2.9)$$

where

$$K(r,\theta,z;r',\theta',w) = -\frac{2}{\pi}\sum_{m=0}^{\infty}\epsilon_m\int_0^{\infty}\frac{\xi K_m'(\xi h)}{I_m'(\xi h)}I_m(\xi r')I_m(\xi r) \times \sin\xi(w-z)\,\mathrm{d}\xi\cos m\theta'\cos m\theta. \quad (2.10)$$

The normal velocity of the fluid on the surface of the ellipsoid is zero. Therefore we have the following boundary integro-differential equation:

$$\frac{\partial^2}{\partial n\partial z}\int_V\frac{f(u,v,w)\,\mathrm{d}\mathbf{v}}{R(\mathbf{x}-\mathbf{u})} + v_0\frac{\partial z}{\partial n} + \frac{\partial}{\partial n}\int_V f(u,v,w)K(r,\theta,z;r',\theta',w)\,\mathrm{d}\mathbf{v} = 0, \qquad (x,y,z)\in S, \quad (2.11)$$

where $\partial/\partial n$ indicates the differentiation in the direction normal to the surface of the ellipsoid. In (2.10) we set $\zeta = \xi h$, then

$$K(r,\theta,z;r',\theta',w) = -\frac{2}{\pi}\sum_{m=0}^{\infty}\epsilon_m\int_0^\infty \frac{\zeta K_m'(\zeta)}{h^2 I_m'(\zeta)} I_m\left(\frac{\zeta}{h}r'\right) I_m\left(\frac{\zeta}{h}r\right) \times \sin\frac{\zeta}{h}(w-z)\,\mathrm{d}\zeta\cos m\theta'\cos m\theta, \tag{2.12}$$

and if we expand (2.12) in terms of $1/h$, we have

$$K(r,\theta,z;r',\theta',w) = -\frac{2}{\pi}\left[\frac{1}{h^3}(w-z)J_2 + \frac{1}{h^5}\left\{\frac{w-z}{4}(r^2+r'^2) - \frac{(w-z)^3}{6}\right\}J_4\right] + O(h^{-6}), \tag{2.13}$$

where J_n is defined by

$$J_n = -\int_0^\infty \frac{K_1(\zeta)}{I_1(\zeta)}\zeta^n\,\mathrm{d}\zeta.$$

So $f(u,v,w)$ is of the form:

$$f(u,v,w) = f_0(u,v,w) + \frac{1}{h^3}f_1(u,v,w) + \frac{1}{h^5}f_2(u,v,w) + \cdots. \tag{2.14}$$

As we stopped at the third term in the infinite expansion in (2.14), the accuracy falls off when $\frac{h}{a^*}$ is near to 1, where $a^* = \max(a,b)$. Here, we assume that the size of the obstacle is small compared to the radius of the tube.

We have zeroth-order solution by solving

$$\frac{\partial^2}{\partial n\partial z}\int_V \frac{f_0(u,v,w)\,\mathrm{d}\mathbf{v}}{R(\mathbf{x}-\mathbf{u})} = -v_0\frac{\partial z}{\partial n}. \tag{2.15}$$

Let (λ,μ,ν) be the usual ellipsoidal coordinates, then the normal derivative of ϕ on the ellipsoid can be expressed as

$$\frac{\partial\phi}{\partial n} = \frac{2}{D_0}\left(\frac{\partial\phi}{\partial\lambda}\right)_{\lambda=0},$$

where

$$D_0^2 = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}.$$

So the suitable solution f_0 of (2.15) is $f_0 =$ constant. We determine f_0 as follows. To do so we use the following formula:

$$\int_V \frac{\mathrm{d}\mathbf{v}}{R(\mathbf{x}-\mathbf{u})} = \pi abc\int_\lambda^\infty \left(1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} - \frac{z^2}{c^2+s}\right)\frac{\mathrm{d}s}{\Delta(s)}, \tag{2.16}$$

where $\Delta(s) = \sqrt{(a^2+s)(b^2+s)(c^2+s)}$ and λ is the greatest root of

$$1 - \frac{x^2}{a^2+\lambda} - \frac{y^2}{b^2+\lambda} - \frac{z^2}{c^2+\lambda} = 0.$$

From (2.15) we obtain

$$f_0 = \frac{v_0}{2\pi(\gamma_0 - 2)}, \tag{2.17}$$

where γ_0 is defined by

$$\gamma_0 = abc\int_0^\infty \frac{\mathrm{d}s}{(c^2+s)\Delta(s)}. \tag{2.18}$$

The solution of the order of h^{-3} is obtained similarly, and we find f_1 as

$$f_1 = \frac{4 f_0 J_2 abc}{\pi(\gamma_0 - 2)3}. \tag{2.19}$$

The problem of the order of h^{-5} is

$$\frac{\partial^2}{\partial n \partial z} \int_V \frac{f_2(u, v, w)\, \mathrm{d}\mathbf{v}}{R(\mathbf{x} - \mathbf{u})} = -\frac{2z}{3c^2 D_0} f_0 J_4 abc \times \left\{ \frac{1}{5}(a^2 + b^2 - 2c^2) + x^2\left(1 + \frac{2c^2}{a^2}\right) + y^2\left(1 + \frac{2c^2}{b^2}\right) - 2z^2 \right\}. \tag{2.20}$$

Then the suitable form of the solution of (2.20) is as follows:

$$f_2(u, v, w) = A_0 + A_1 u^2 + A_2 v^2 + A_3 w^2. \tag{2.21}$$

Proceeding as before we find

$$A_0 = \frac{J_4 f_0 (a^2 + b^2 - 2c^2) abc}{15\pi(\gamma_0 - 2)}.$$

To find A_1, A_2 and A_3, we utilize the following formulae found in Ferrers [10]

$$\int_V \frac{u^2 \mathrm{d}\mathbf{v}}{R(\mathbf{x} - \mathbf{u})} = \pi abc \int_\lambda^\infty \left\{ \frac{a^2 s}{4(a^2 + s)} \omega^2(s) - \frac{a^4 x^2}{(a^2 + s)^2} \omega(s) \right\} \frac{\mathrm{d}s}{\Delta(s)}$$

$$\int_V \frac{v^2 \mathrm{d}\mathbf{v}}{R(\mathbf{x} - \mathbf{u})} = \pi abc \int_\lambda^\infty \left\{ \frac{b^2 s}{4(b^2 + s)} \omega^2(s) - \frac{b^4 y^2}{(b^2 + s)^2} \omega(s) \right\} \frac{\mathrm{d}s}{\Delta(s)}$$

$$\int_V \frac{w^2 \mathrm{d}\mathbf{v}}{R(\mathbf{x} - \mathbf{u})} = \pi abc \int_\lambda^\infty \left\{ \frac{c^2 s}{4(c^2 + s)} \omega^2(s) - \frac{c^4 z^2}{(c^2 + s)^2} \omega(s) \right\} \frac{\mathrm{d}s}{\Delta(s)},$$

where

$$\omega(s) = \frac{x^2}{a^2 + s} + \frac{y^2}{b^2 + s} + \frac{z^2}{c^2 + s} - 1.$$

After a long computation we obtain the following simultaneous equation for A_1, A_2 and A_3:

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = -\frac{f_0 J_4}{3\pi c^2} \begin{pmatrix} 1 + 2c^2/a^2 \\ 1 + 2c^2/b^2 \\ -2 \end{pmatrix}, \tag{2.22}$$

where

$$b_{11} = I_{2,0,0} a^2 \left(\frac{1}{a^2} + \frac{1}{2c^2}\right) - \frac{I_{1,0,0}}{2c^2} - I_{2,0,1} a^2 \left(\frac{5}{2} + \frac{c^2}{a^2} + \frac{a^2}{c^2}\right) + \frac{I_{1,0,1}}{2} + \frac{2}{abc^3}, \tag{2.23}$$

and $b_{12}, \ldots, b_{33}$ are listed in the appendix. In (2.23) $I_{\ell,m,n}$ is defined by

$$I_{\ell,m,n} = \int_0^\infty \frac{1}{(a^2 + s)^\ell (b^2 + s)^m (c^2 + s)^n} \frac{\mathrm{d}s}{\Delta(s)}. \tag{2.24}$$

To evaluate (2.24) we let

$$s = (a^2 - c^2)\mathrm{sn}^{-2} u$$

and use the following identities for the Jacobian elliptic functions:

$$k^2\, \mathrm{sn}^2 u + \mathrm{dn}^2 u = 1$$

and

$$k = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \qquad k'^2 = 1 - k^2.$$

So

$$\int_0^\infty \frac{1}{(a^2+s)^\ell (b^2+s)^m (c^2+s)^n} \frac{\mathrm{d}s}{\Delta(s)} = \frac{2}{(a^2-c^2)^{\ell+m+n+\frac{1}{2}}} \int_0^F \frac{\mathrm{sn}^{2\ell+2m+2n} u \,\mathrm{d}u}{\mathrm{dn}^{2m} u \,\mathrm{cn}^{2n} u}, \tag{2.25}$$

where

$$F = \int_0^\theta (1-k^2\sin^2 u)^{-\frac{1}{2}}\,\mathrm{d}u, \qquad \theta = \sin^{-1}\left(\frac{\sqrt{a^2-c^2}}{a}\right). \tag{2.26}$$

The integral on the right-hand side of (2.25) is

$$L_{\ell,m,n} = \int_0^F \frac{\mathrm{sn}^{2\ell+2m+2n} u\,\mathrm{d}u}{\mathrm{dn}^{2m} u\,\mathrm{cn}^{2n} u} = \frac{1}{k'^{2\ell+2m+2n}} \sum_{j=0}^{\ell+m+n} (-1)^j \binom{\ell+n+m}{j} \times \int_0^F \mathrm{dn}^{2\ell+2n-2j} u\,\mathrm{nc}^{2n-2j} u\,\mathrm{d}u. \tag{2.27}$$

Further

$$L_{\ell,m,n} = \frac{1}{k'^{2\ell+2m+2n}} \left[\sum_{j=0}^{\ell+n} (-1)^j \binom{\ell+n+m}{j} \int_0^F (k'^2 + k^2\mathrm{cn}^2 u)^{\ell+n-j} \mathrm{nc}^{2n-2j} u\,\mathrm{d}u + \sum_{j=\ell+n+1}^{\ell+m+n} (-1)^j \binom{\ell+n+m}{j} \int_0^F \mathrm{nd}^{2j-2\ell-2n} u \left(\frac{\mathrm{dn}^2 u - k'^2}{k^2}\right)^{j-n} \mathrm{d}u \right]. \tag{2.28}$$

Expanding the powered terms by using the binomial expansion, we find

$$L_{\ell,m,n} = \frac{1}{k'^{2\ell+2m+2n}} \left[\sum_{j=0}^{\ell+n} \sum_{i=0}^{\ell+n-j} (-1)^j \binom{\ell+n+m}{j} \binom{\ell+n-j}{i} k'^{2i} \times k^{2\ell+2n-2j-2i} C_{2\ell-2i} + \sum_{j=\ell+n+1}^{\ell+m+n} \sum_{i=0}^{j-n} (-1)^{j+i} \binom{\ell+n+m}{j} \binom{j-n}{i} k'^{2i} k^{2n-2k} G_{2\ell-2i} \right], \tag{2.29}$$

where

$$C_{2n} = \int_0^F \mathrm{cn}^{2n} u\,\mathrm{d}u, \qquad G_{2n} = \int_0^F \mathrm{dn}^{2n} u\,\mathrm{d}u.$$

We have the following reduction formula for C_{2n} in Byrd and Friedman [11, p 194]:

$$C_{2n+2} = \frac{2n(2k^2-1)C_{2n} + (2n-1)k'^2 C_{2n-2} + \mathrm{sn}\,F\,\mathrm{dn}\,F\,\mathrm{cn}^{2n-1}\,F}{(2n+1)k^2}, \tag{2.30}$$

if $\ell - i < 0$, we find $C_{-2n} = D_{2n}$ where

$$D_{2n+2} = \frac{(2n-1)k^2 D_{2n-2} + 2n(1-2k^2)D_{2n} + \mathrm{tn}\,F\,\mathrm{dn}\,F\,\mathrm{nc}^{2n}\,F}{(2n+1)k'^2}. \tag{2.31}$$

Also we have the following reduction formula for G_{2n}:

$$G_{2n+2} = \frac{k^2\,\mathrm{dn}^{2n-1} F\,\mathrm{sn}\,F\,\mathrm{cn}\,F + (1-2n)k'^2 G_{2n-2} + 2n(2-k^2)G_{2n}}{(2n+1)}, \tag{2.32}$$

if $\ell - i < 0$, we find $G_{-2n} = I_{2n}$ where

$$I_{2n+2} = \frac{2n(2-k^2)I_{2n} + (1-2n)I_{2n-2} - k^2\,\mathrm{sn}\,F\,\mathrm{cn}\,F\mathrm{nd}^{2n+1}F}{(2n+1)k'^2}. \tag{2.33}$$

Thus, finally, we see that one needs the following starting values for evaluating the general terms of C_{2n}, D_{2n}, G_{2n} and I_{2n}:

$$C_0 = D_0 = F, \qquad C_2 = \frac{1}{k^2}[E - k'^2 F], \qquad D_2 = \frac{1}{k'^2}[k'^2 F - E + \operatorname{dn} F \operatorname{tn} F],$$

where E is the elliptic integral of the second kind of modulus k and argument θ given in (2.26) and

$$G_0 = I_0 = F, \qquad G_2 = E, \qquad I_2 = \frac{1}{k'^2}[E - k^2 \operatorname{sn} F \operatorname{cd} F]$$

and

$$\operatorname{sn} F = \frac{\sqrt{a^2 - c^2}}{a}, \qquad \operatorname{cn} F = \frac{c}{a}, \qquad \operatorname{dn} F = \frac{b}{a}.$$

The final solution is given by (2.9) with $f(u, v, w)$ replaced by (2.14) where f_0, f_1, f_2 are now (2.17), (2.19) and (2.21), respectively. The kernel is substituted by (2.13). Therefore as h approaches to infinity, the potential function ϕ is now

$$\phi = \frac{\partial}{\partial z} \int_V \frac{f_0(u, v, w)\, \mathrm{d}\mathbf{v}}{R(\mathbf{x} - \mathbf{u})} + v_0 z = \frac{v_0 zabc}{2 - \gamma_0} \int_\lambda^\infty \frac{\mathrm{d}s}{(c^2 + s)\Delta(s)} + v_0 z, \qquad (2.34)$$

This solution is in complete agreement with that in Lamb [10, p 153]. When the ellipsoid reduces to a sphere, the solution greatly simplifies and it is recorded as follows:

$$\phi = \left(\frac{v_0 a^3}{2}\frac{\cos\varphi}{\rho^2} + v_0 \rho \cos\varphi\right)\left(1 - \frac{1}{h^3}\frac{a^3}{\pi} J_2\right) - \frac{1}{h^5} v_0 (3\cos\varphi - 5\cos^3\varphi)\left(\frac{a^{10}}{4\rho^4} + \frac{a^3\rho^3}{3}\right)\frac{J_4}{4\pi}, \qquad (2.35)$$

where φ is the azimuthal angle and ρ is the radius in spherical coordinates and a is the radius of the sphere.

The stream function is as follows:

$$\psi = \left(-v_0 \frac{a^3}{2}\frac{\sin^2\varphi}{\rho} + \frac{v_0}{2}\rho^2 \sin^2\varphi\right)\left(1 - \frac{1}{h^3}\frac{a^3}{\pi} J_2\right) + \frac{1}{h^5} v_0 (\sin^2\varphi - 5\cos^2\varphi \sin^2\varphi)\left(\frac{a^{10}}{\rho^3} - a^3\rho^4\right)\frac{J_4}{16\pi}, \qquad (2.36)$$

As h tends to infinity, the potential function and the stream function, given by (2.35) and (2.36), respectively, completely agree with those found in Streeter [13, p 67]. Also we can immediately see from (2.35) and (2.36) that the normal velocity and stream function are zero on the surface of the sphere which also confirms the correctness of our solution.

3. Viscous flow about a spheroid

In this section we consider the Stokes problem for spheroids when the fluid is viscous. The fluid velocity $\mathbf{u}$ and the pressure p satisfy the Stokes equation and the continuity equation:

$$\nabla p = \mu \nabla^2 \mathbf{u}, \qquad (3.1)$$

$$\nabla \cdot \mathbf{u} = 0, \qquad (3.2)$$

where μ is the coefficient of viscosity. Let $\mathbf{u} = (u_x, u_y, u_z)$ be the velocity components in Cartesian coordinates. If we choose the velocity components as

$$u_x = 2B_x - \frac{\partial \Phi}{\partial x}, \qquad u_y = 2B_y - \frac{\partial \Phi}{\partial y}, \qquad u_z = 2B_z - \frac{\partial \Phi}{\partial z}, \qquad (3.3)$$

where Φ is defined by

$$\Phi = B_0 + xB_x + yB_y + zB_z$$

with B_0, B_x, B_y, B_z being the harmonic functions, we see that (3.1) and (3.2) are satisfied by

$$p = 2\mu\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right). \tag{3.4}$$

The suitable functions for the problem are

$$B_x = \int_{-\infty}^{\infty} A(\xi)I_1(\xi r)\,\mathrm{e}^{-\mathrm{i}\xi z}\,\mathrm{d}\xi\,\cos\theta, \tag{3.5}$$

$$B_y = \int_{-\infty}^{\infty} A(\xi)I_1(\xi r)\,\mathrm{e}^{-\mathrm{i}\xi z}\,\mathrm{d}\xi\,\sin\theta, \tag{3.6}$$

$$B_z = \left\{1 - \frac{1}{2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\right\}\int_V \frac{a(\mathbf{u})\,\mathrm{d}\mathbf{v}}{R(\mathbf{x}-\mathbf{u})}, \tag{3.7}$$

$$B_0 = \frac{\partial}{\partial z}\int_V \frac{b(\mathbf{u})\,\mathrm{d}\mathbf{v}}{R(\mathbf{x}-\mathbf{u})} + \int_{-\infty}^{\infty} B(\xi)hI_0(\xi r)\,\mathrm{e}^{-\mathrm{i}\xi z}\,\mathrm{d}\xi, \tag{3.8}$$

where $a(\mathbf{u})$ and $b(\mathbf{u})$ are unknowns to be determined. Let (u_r, u_θ, u_z) be the velocity components in cylindrical coordinates. The velocity at the tube wall is zero

$$u_r = 0, \tag{3.9a}$$

$$u_\theta = 0, \tag{3.9b}$$

$$u_z = 0. \tag{3.9c}$$

Boundary condition (3.9*a*) can be written in an alternative form as

$$\mathcal{F}[u_r(h,\theta,z); z \to \xi] = 0,$$

where $\mathcal{F}$ means the Fourier transform. If $a(\mathbf{u})$ and $b(\mathbf{u})$ are axisymmetric functions, we obtain the following relation for solving unknown $A(\xi)$, and $B(\xi)$ after using relations (2.4) and (2.6)

$$\xi hI_2(\xi h)A(\xi) + \xi hI_1(\xi h)B(\xi) = \frac{1}{\pi\mathrm{i}}\left(\xi I(\xi)|\xi|K_1(|\xi|h) + \frac{\partial}{\partial\xi}\{J(\xi)|\xi|K_1(|\xi|h)\}\right), \tag{3.10}$$

where $I(\xi)$ and $J(\xi)$ are defined by

$$I(\xi) = \int_V b(\mathbf{u})I_0(\xi r')\,\mathrm{e}^{\mathrm{i}\xi w}\,\mathrm{d}\mathbf{v} \qquad \text{and} \qquad J(\xi) = \int_V a(\mathbf{u})f(\xi, r', w)\,\mathrm{e}^{\mathrm{i}\xi w}\mathrm{d}\mathbf{v}$$

with

$$f(\xi, r', w) = \frac{3}{2}I_0(\xi r') + \frac{\xi r'}{2}I_1(\xi r') + \frac{1}{2}\mathrm{i}w\xi I_0(\xi r').$$

Condition (3.9*b*) is automatically satisfied by this choice of functions. Condition (3.9*c*) can be alternatively written as

$$\mathcal{F}[u_z(h,\theta,z); z \to \xi] = 0$$

from which we obtain another relation to solve unknown $A(\xi)$ and $B(\xi)$:

$$\xi h I_1(\xi h)A(\xi) + \xi h I_0(\xi h)B(\xi)$$
$$= -\frac{1}{\pi \mathrm{i}}\left\{\xi^2 I(\xi)K_0(|\xi|h) + \left(2+\xi\frac{\partial}{\partial\xi}\right)\{J(\xi)|\xi|K_0(|\xi|h)\}\right\} + \mathrm{i}\frac{v_0}{2}\delta(\xi), \qquad (3.11)$$

where $\delta(\xi)$ is the Dirac delta function and we have used the known relation

$$\frac{1}{\pi}\int_0^\infty \cos\xi z\,\mathrm{d}z = \delta(\xi).$$

Therefore if we solve (3.10) and (3.11) simultaneously for $A(\xi)$ and $B(\xi)$, we obtain the following equations:

$$A(\xi) = -\frac{1}{\pi\mathrm{i}}\left[\int_V a(\mathbf{u})\left\{\frac{F(\xi,r',w)}{\Delta(\xi h)} + f(\xi,r',w)\left(G(|\xi|h) - \frac{2}{\Delta(\xi h)}\right)\right\}\mathrm{e}^{\mathrm{i}\xi w}\,\mathrm{d}\mathbf{v}\right.$$
$$\left. + \frac{\xi^2}{\Delta(\xi h)}\int_V b(\mathbf{u})I_0(\xi r')\,\mathrm{e}^{\mathrm{i}\xi w}\,\mathrm{d}\mathbf{v}\right] - \frac{v_0}{2}\frac{\delta(\xi)\xi h I_1(\xi h)}{\mathrm{i}\Delta(\xi h)}, \qquad (3.12)$$

where

$$\Delta(\zeta) = -\zeta^2\{I_2(\zeta)I_0(\zeta) - I_1^2(\zeta)\}, \qquad G(|\zeta|) = \frac{K_0(|\zeta|)}{I_0(\zeta)} - \frac{I_1(\zeta)\zeta}{I_0(\zeta)\Delta(\zeta)}$$

$$F(\xi,r',w) = I_0(\xi r')\left\{3 + \frac{\xi^2}{2}(r'^2 - w^2)\right\} + \frac{5}{2}\xi r' I_1(\xi r') + \mathrm{i}\xi w\{3I_0(\xi r') + \xi r' I_1(\xi r')\}$$

and

$$B(\xi)h = -\frac{1}{\pi\mathrm{i}}\left[\int_V a(\mathbf{u})\left\{\frac{1}{\xi}G(|\xi|h)F(\xi,r',w) + \xi h^2\frac{f(\xi,r',w)}{\Delta(\xi h)}\right\}\mathrm{e}^{\mathrm{i}\xi w}\,\mathrm{d}\mathbf{v}\right.$$
$$\left. + G(|\xi|h)\xi\int_V b(\mathbf{u})I_0(\xi r')\,\mathrm{e}^{\mathrm{i}\xi w}\,\mathrm{d}\mathbf{v}\right] + \frac{v_0}{2}\frac{\delta(\xi)\xi h^2 I_2(\xi h)}{\mathrm{i}\Delta(\xi h)}. \qquad (3.13)$$

The velocity components (u_x, u_y, u_z) are zero on the surface of the ellipsoid. Thus if we substitute the values of $A(\xi)$ and $B(\xi)$ given by (3.12) and (3.13) into B_x, B_y and B_0 in (3.5), (3.6) and (3.8), we obtain the following three conditions:

$$u_x = \frac{\partial\Psi}{\partial x} + x\Phi = 0, \qquad (x,y,z)\in S \qquad (3.14a)$$

$$u_y = \frac{\partial\Psi}{\partial y} + y\Phi = 0, \qquad (x,y,z)\in S, \qquad (3.14b)$$

where

$$\Psi = -\frac{\partial}{\partial z}\int_V \frac{b(\mathbf{u})\,\mathrm{d}\mathbf{v}}{R(\mathbf{x}-\mathbf{u})} - z\mathcal{L}\int_V \frac{a(\mathbf{u})\,\mathrm{d}\mathbf{v}}{R(\mathbf{x}-\mathbf{u})},$$

$$\Phi = \frac{2}{\pi}\int_V a(\mathbf{u})\int_0^\infty\left[\mathcal{F}_1(\xi,r',w,z)\left\{G(\zeta)\frac{I_1(\xi r)}{r} + \frac{\xi I_2(\xi r)}{\Delta(\zeta)}\right\}\right.$$
$$\left. + \mathcal{F}_2(\xi,r',w,z)\left\{\left(G(\zeta) - \frac{2}{\Delta(\zeta)}\right)\xi I_2(\xi r) + \frac{\zeta^2}{\Delta(\zeta)}\frac{I_1(\xi r)}{r}\right\}\right]\mathrm{d}\xi\,\mathrm{d}\mathbf{v}$$
$$+ \frac{2}{\pi}\int_V b(\mathbf{u})\int_0^\infty\left\{G(\zeta)\frac{I_1(\xi r)}{r} + \frac{\xi I_2(\xi r)}{\Delta(\zeta)}\right\}I_0(\xi r')\xi^2\sin\xi(w-z)\,\mathrm{d}\xi\,\mathrm{d}\mathbf{v}, \qquad (3.15)$$

where

$$\mathcal{F}_1(\xi, r', w, z) = \text{Re } F(\xi, r', w) \sin \xi(w - z) + \text{Im} F(\xi, r', w) \cos \xi(w - z)$$
$$\mathcal{F}_2(\xi, r', w, z) = \text{Re } f(\xi, r', w) \sin \xi(w - z) + \text{Im} f(\xi, r', w) \cos \xi(w - z)$$
$$\mathcal{L} = 1 - \frac{1}{2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$

and $\zeta = \xi h$.

$$u_z = 2\mathcal{L}\int_V \frac{a(\mathbf{u})\,\mathrm{d}\mathbf{v}}{R(\mathbf{x}-\mathbf{u})} + \frac{\partial}{\partial z}\left[\frac{2}{\pi}\int_V a(\mathbf{u})\int_0^\infty \frac{1}{\xi}\left(\mathcal{F}_1(\xi, r', w, z)\left\{G(\zeta)I_0(\xi r) + \frac{\xi r I_1(\xi r)}{\Delta(\zeta)}\right\} + \mathcal{F}_2(\xi, r', w, z)\left\{\left(G(\zeta) - \frac{2}{\Delta(\zeta)}\right)\xi r I_1(\xi r) + \frac{\zeta^2}{\Delta(\zeta)}I_0(\xi r)\right\}\right)\mathrm{d}\xi\,\mathrm{d}\mathbf{v} + \frac{2}{\pi}\int_V b(\mathbf{u})\int_0^\infty\left\{G(\zeta)I_0(\xi r) + \frac{\xi r I_1(\xi r)}{\Delta(\zeta)}\right\}I_0(\xi r')\xi \sin\xi(w-z)\,\mathrm{d}\xi\,\mathrm{d}\mathbf{v} - \frac{\partial}{\partial z}\int_V \frac{b(\mathbf{u})\,\mathrm{d}\mathbf{v}}{R(\mathbf{x}-\mathbf{u})} - z\mathcal{L}\int_V \frac{a(\mathbf{u})\,\mathrm{d}\mathbf{v}}{R(\mathbf{x}-\mathbf{u})}\right] + v_0\left(1 - \frac{r^2}{h^2}\right) = 0, \quad (x, y, z) \in S. \tag{3.16}$$

From conditions (3.14*a*) and (3.14*b*) we get the same results. Thus we have two equations with two unknowns to solve. It is not convenient for us to solve these equations using the method in section 2; a numerical method is more useful, so at this point we will employ the Galerkin method.

Of interest here is the circumstance where the radius of the cylinder tends to infinity. All terms involving h vanish, and pertinent functions for the solution are constants. So

$$a(\mathbf{u}) = a_1, \qquad b(\mathbf{u}) = b_1 \quad \text{(say)}.$$

Then from (2.16) we find that

$$\Psi = -\pi z a_1 \chi + 2\pi z b_1 \gamma,$$

where

$$\chi = abc\int_\lambda^\infty \frac{\mathrm{d}s}{\Delta(s)}, \qquad \gamma = abc\int_\lambda^\infty \frac{\mathrm{d}s}{(c^2+s)\Delta(s)}. \tag{3.17}$$

Conditions (3.14) require

$$\left[-a_1\frac{\mathrm{d}\chi}{\mathrm{d}\lambda} + 2b_1\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}\right]_{\lambda=0} = 0 \qquad \text{or} \qquad -a_1 + 2\frac{b_1}{c^2} = 0.$$

With the help of this relation, the condition $u_z = 0$ reduces to

$$v_0 + (a_1\chi_0 + 2b_1\gamma_0)\pi = 0, \tag{3.18}$$

where the suffix denotes that the lower limit in the integrals (3.17) is to be replaced by zero. Hence

$$b_1 = \frac{1}{2}a_1 c^2, \qquad a_1 = -\frac{v_0}{\pi(\chi_0 + \gamma_0 c^2)}.$$

This is in agreement with Lamb [12, p 605].

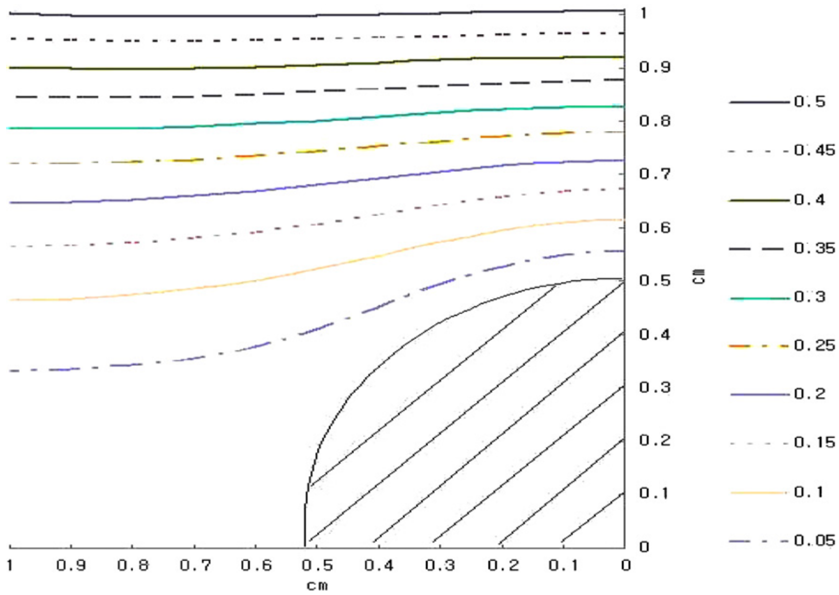

Figure 1. Streamlines (ψ/v_0 (cm)) for flow around a sphere inside a cylinder of radius 1 cm. Radius of sphere 0.5 cm.

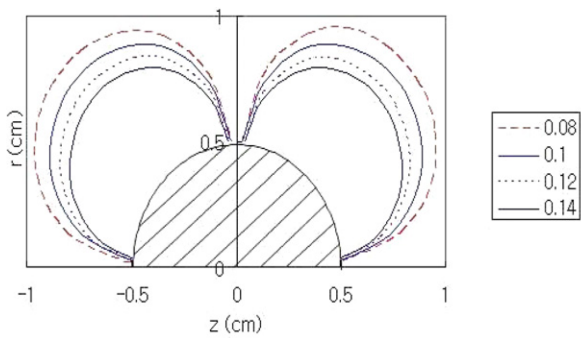

Figure 2. Non-dimensional velocity component v_r/v_0 for a sphere of radius of 0.5 cm in a tube of radius 1 cm.

4. Numerical examples

In this section, we present some numerical examples.

Stream lines using (2.36) are shown in figure 1. Comparing with figure 1 in Lai [14], the agreement between the two is excellent. The values of J_2 and J_4 are found to be -2.504 and -3.770, respectively.

In figures 2 and 3, we show the equi-velocity lines v_r/v_0, v_z/v_0 respectively, for the sphere. In figures 1–3, the radius of the sphere is 0.5 cm and that of the tube is 1 cm.

We can calculate the pressure on the surface of the ellipsoidal obstacle in the inviscid fluid by using Bernoulli's equation. Thus, the difference in pressures is

$$\frac{p_\infty - p}{\rho^*} = \frac{1}{2}\left(|\nabla\phi|^2 - v_0^2\right),$$

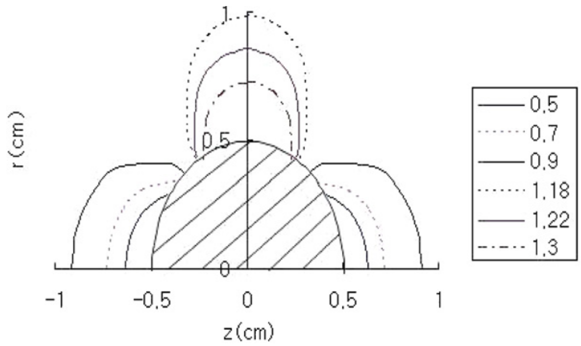

Figure 3. Non-dimensional velocity component v_z/v_0 for a sphere of radius of 0.5 cm in a tube of radius 1 cm.

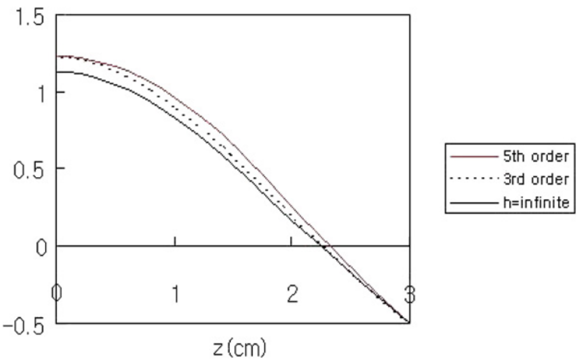

Figure 4. Non-dimensional pressure $(p_\infty - p)/\rho v_0^2$ for an ellipsoid $a = 5$ cm, $b = 4$ cm and $c = 3$ cm in a tube of radius of 12 cm in an inviscid fluid.

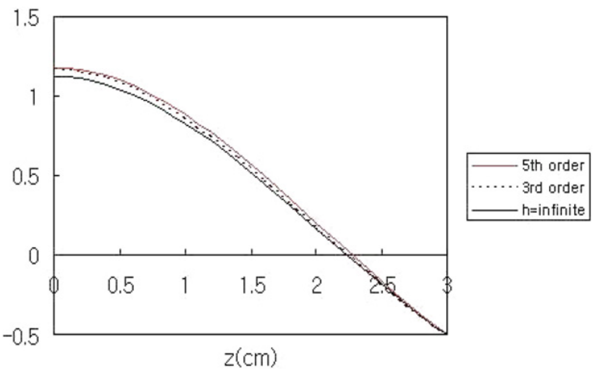

Figure 5. Non-dimensional pressure $(p_\infty - p)/\rho v_0^2$ for an ellipsoid $a = 5$ cm, $b = 4$ cm and $c = 3$ cm in a tube of radius of 15 cm in an inviscid fluid.

where ρ^* is the density of the fluid. For numerical example we take $a = 5$ cm, $b = 4$ cm and $c = 3$ cm. So $k = 3/4, \theta = \sin^{-1}(0.75) = 48.59^\circ$, $E = 0.859$ and $F = 1.0$. In figures 4 and 5, we plotted $(p_\infty - p)/\rho v_0^2$ versus z for $h = 12$ cm and 15 cm, respectively, when

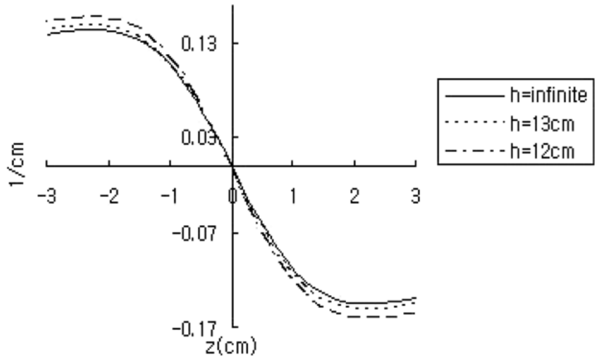

Figure 6. Pressure $(p_\infty - p)/2\mu v_0$ on the surface of a spheroid ($a = b = 5$ cm, $c = 3$ cm) in a viscous fluid.

$x = 0$. The lower curve is a third-order $((1/h)^3)$, and upper curve is a fifth-order$((1/h)^5)$ solution.

As a final numerical example, we have calculated the pressure for the viscous fluid. In figure 6 we present the variation of $(p_\infty - p)/2\mu v_0$ with respect to z on the surface of the spheroid when $x = 0$. Here $a = b = 5$ cm and $c = 3$ cm.

5. Conclusion

We have presented the analytical solution of an irrotational, inviscid fluid flow around a triaxial ellipsoid in a circular tube. Also, we dealt with a numerical solution for a viscous fluid when the ellipsoid is axisymmetric. This solution agrees with the published accounts when the radius of the cylinder approaches to infinity. The solution, when the radius of the cylinder is finite, is compared with the result obtained by another investigator when the ellipsoid reduces to a sphere. The excellent agreements between the solutions support the correctness of our solution. The solution for the triaxial ellipsoid is used to compute the pressure on the surface of the obstacle. Judging from the results for the infinite medium, the solution also appears to be correct.

Appendix

$$b_{12} = I_{1,1,0} b^2\left(\frac{1}{2c^2} + \frac{1}{a^2}\right) - I_{0,1,0}\frac{b^2}{2a^2c^2} - I_{1,1,1} b^2 c^2\left(\frac{1}{2c^2} + \frac{1}{a^2}\right) + I_{0,1,1}\frac{b^2}{2a^2},$$

$$b_{13} = I_{1,0,1} c^2\left(\frac{1}{2c^2} + \frac{1}{a^2}\right) - I_{0,0,1}\frac{1}{2a^2} - I_{1,0,2} 3c^4\left(\frac{1}{2c^2} + \frac{1}{a^2}\right) + I_{0,0,2}\frac{3c^2}{2a^2},$$

$$b_{21} = I_{1,1,0} a^2\left(\frac{1}{2c^2} + \frac{1}{b^2}\right) - I_{1,0,0}\frac{a^2}{2b^2c^2} - I_{1,1,1} a^2 c^2\left(\frac{1}{2c^2} + \frac{1}{b^2}\right) + I_{1,0,1}\frac{a^2}{2b^2},$$

$$b_{22} = I_{0,2,0} b^2\left(\frac{1}{2c^2} + \frac{1}{b^2}\right) - I_{0,1,0}\frac{1}{2c^2} - I_{0,2,1} b^2\left(\frac{5}{2} + \frac{b^2}{c^2} + \frac{c^2}{b^2}\right) + I_{0,1,1}\frac{1}{2} + \frac{2}{abc^3},$$

$$b_{23} = I_{0,1,1} c^2\left(\frac{1}{2c^2} + \frac{1}{b^2}\right) - I_{0,0,1}\frac{1}{2b^2} - I_{0,1,2} 3c^4\left(\frac{1}{2c^2} + \frac{1}{b^2}\right) + I_{0,0,2}\frac{3c^2}{2b^2},$$

$$b_{31} = I_{1,0,1}\frac{3a^2}{2c^2} - I_{1,0,0}\frac{a^2}{2c^4} - I_{1,0,2}\frac{3a^2}{2} + I_{1,0,1}\frac{a^2}{2c^2},$$

$$b_{32} = I_{0,1,1}\frac{3b^2}{2c^2} - I_{0,1,0}\frac{b^2}{2c^4} - I_{0,1,2}\frac{3b^2}{2} + I_{0,1,1}\frac{b^2}{2c^2},$$

$$b_{33} = I_{0,0,2}\frac{3}{2} - I_{0,0,1}\frac{1}{2c^2} - I_{0,0,3}\frac{15c^2}{2} + I_{0,0,2}\frac{3}{2} + \frac{2}{abc^3}.$$

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